

COXETER-CHEIN LOOPS

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ABSTRACT. In 1974 Orin Chein discovered a new family of Moufang loops which are now called Chein loops. Such a loop can be created from any group W together with \mathbb{Z}_2 by a variation on a semi-direct product. We study these loops in the case where W is a Coxeter group and show that it has what we call a Chein-Coxeter system, a small set of generators of order 2, together with a set of relations closely related to the Coxeter relations and Chein relations. As a result we are able to give amalgam presentations for Coxeter-Chein loops. This is to our knowledge the first such presentation for a Moufang loop.

Keywords: Coxeter groups, amalgams, Moufang loops, generators and relations.

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1. INTRODUCTION

All Moufang loops considered in this article are finite. However, the definitions below may still hold for loops of infinite order. A *quasigroup* is a nonempty set S with a closed binary operation,

$$(x, y) \mapsto x \cdot y,$$

such that:

- (a) $a \cdot x = b$ determines a unique element $x \in S$ given $a, b \in S$; and
- (b) $b = y \cdot a$ determines a unique element $y \in S$ given $a, b \in S$.

A *loop* is a quasigroup L with a 2-sided identity element. It was Ruth Moufang who introduced what we now call the Moufang identity,

$$(m1) \quad z(x(yx)) = ((zx)y)x \quad \text{for all } x, y, z \in L$$

One can show (see Lemma 3.1 of [2]) that in any loop (m1) is actually equivalent to:

$$(m2) \quad x(y(xz)) = ((xy)x)z$$

$$(m3) \quad (xy)(zx) = x(yz)x,$$

for all $x, y, z \in L$.

Definition 1.1. A *Moufang loop* is a loop, L , that satisfies the Moufang identities.

Definition 1.2. Given a group G the *Chein loop* [3] over G , denoted $M(G, 2)$ is the set $G \uplus Gu$, where u is some formal element not in G together with a binary operation that extends that of G and satisfies

$$\begin{aligned} \text{(c1)} \quad & g_1(g_2u) = (g_2g_1)u, \\ \text{(c2)} \quad & (g_1u)g_2 = (g_1g_2^{-1})u, \\ \text{(c3)} \quad & (g_1u)(g_2u) = g_2^{-1}g_1, \end{aligned}$$

for all $g_1, g_2 \in G$.

Definition 1.3. A *Coxeter diagram* over the index set $I = \{1, 2, \dots, n\}$ is a symmetric matrix $\mathbf{M} = (m_{ij})_{i,j \in I}$ with entries in $\mathbb{N}_{\geq 1} \cup \{\infty\}$ such that $m_{ii} = 1$ for all $i \in I$.

A *Coxeter system with diagram* \mathbf{M} is a pair $(W, \{s_i\}_{i \in I})$ where W is a group and $\{s_i\}_{i \in I}$ is a set of generators with defining relations

$$(s_i s_j)^{m_{ij}} = 1 \text{ for all } i, j \in I.$$

We call \mathbf{M} and W *spherical* if W is finite. For $2 \leq k \leq n$, we call \mathbf{M} and W *k-spherical* if every subdiagram of \mathbf{M} induced on k nodes is spherical.

Definition 1.4. Let \mathbf{M} be a Coxeter diagram over the index set $I = \{1, \dots, n\}$. The *Coxeter-Chein loop with diagram* \mathbf{M} , denoted $C(\mathbf{M})$, is the Chein loop $M(W, 2)$ where $(W, \{s_i\}_{i \in I})$ is a Coxeter system with diagram \mathbf{M} .

From now on we shall fix a Coxeter system $(W, \{s_i\}_{i \in I})$ with spherical diagram \mathbf{M} over the index set $I = \{1, 2, \dots, n\}$. We also assume that u and $C = C(\mathbf{M}) = M(W, 2)$ are as in Definition 1.4.

Before we continue working with Coxeter groups, we shall first prove an open problem which was originally proposed by Petr Vojtěchovský in 2003.

2. MINIMAL PRESENTATIONS FOR $M(G, 2)$

Theorem 2.1. *Let T be a group generated by a set $S = \{s_i\}_{i \in J}$ of elements. Here J can be any set. Then, $M(T, u)$ is the Moufang loop generated by $\{s_i\}_{i \in J} \cup \{u\}$ whose defining relations are those of T , together with the relations $((g_1 g_2)u)^2 = 1$ where $g_1, g_2 \in \{s_i\}_{i \in J} \cup \{e\}$.*

The significance of this theorem is that the Chein relations are not a priori required to hold for all $g_1, g_2 \in T$.

We shall prove Theorem 2.1 in a series of lemmas. We shall assume that L is a Moufang loop containing T and an element u such that the relations $((g_1 g_2)u)^2 = 1$ hold for $g_1, g_2 \in \{s_i\}_{i \in J} \cup \{e\}$. Note that this is equivalent to saying that $u(g_1 g_2)u = (g_1 g_2)^{-1}$ for all $g_1, g_2 \in \{s_i\}_{i \in J} \cup \{e\}$.

Lemma 2.2. *The Chein relations (c1), (c2), and (c3) hold for any $g_1, g_2 \in \{s_i\}_{i \in J}$.*

Proof We have

$$\begin{aligned} s_i(s_j u) &= u(u(s_i(uss_j u))) \\ &= u((us_i u)(us_j u)) \quad (\text{m2}) \\ &= u(s_i^{-1} s_j^{-1}) \\ &= uu(s_j s_i)u \\ &= (s_j s_i)u, \end{aligned}$$

$$\begin{aligned} (s_i u)s_j &= (((s_i u)s_j)u)u \\ &= (s_i(us_j u))u \quad (\text{m1}) \\ &= (s_i s_j^{-1})u, \end{aligned}$$

and

$$\begin{aligned} (s_i u)(s_j u) &= (us_i^{-1})(s_j u) \\ &= u(s_i^{-1} s_j)u \quad (\text{m3}) \\ &= s_j^{-1} s_i. \end{aligned}$$

□

Lemma 2.3. *For any $i, j \in J$ we have*

- (a) $s_i u = us_i^{-1}$
- (b) $s_i(s_j u) = (s_j s_i)u$
- (c) $(s_i u)s_j = (s_i s_j^{-1})u$
- (d) $(s_i u)(s_j u) = s_j^{-1} s_i$
- (e) $(us_i)s_j = s_j^{-1}(us_i)$

Proof (a) follows from (c3) taking $g_1 = e$ and $g_2 = s_i$. (b), (c), and (d) are just the Chein relations which hold from Lemma 2.2. (e) We use (a), (c), (d) and (a) again to see that $(us_i)s_j = (s_i^{-1}u)s_j = (s_i^{-1}s_j^{-1})u = s_j^{-1}(s_i^{-1}u) = s_j^{-1}(us_i)$. □

Lemma 2.4. *For any $i_1, \dots, i_k \in J$ we have $u(s_{i_1} \cdots s_{i_k}) = s_{i_1}^{-1}(u(s_{i_2} \cdots s_{i_k}))$*

Proof The following equalities are equivalent. Write $w = s_{i_2} \cdots s_{i_k}$.

$$\begin{aligned} u(s_{i_1} w) &= s_{i_1}^{-1}(uw) \\ s_{i_1} w &= u(s_{i_1}^{-1}(uw)) \quad \text{mult. by } u \\ s_{i_1} w &= ((us_{i_1}^{-1})u)w \quad (\text{m2}) \\ s_{i_1} &= ((us_{i_1}^{-1})u) \quad \text{mult. by } w^{-1} \\ s_{i_1} &= s_{i_1} \quad \text{Lemma 2.3 (a)} \end{aligned}$$

□

Lemma 2.5. *For any $i_1, \dots, i_k \in J$ we have $u(s_{i_1} \cdots s_{i_k})u = s_{i_k}^{-1} \cdots s_{i_1}^{-1}$. In other words, for all $w \in T$, $uwu = w^{-1}$.*

Proof For $k = 1$ this is true by (a) of Lemma 2.3. For $k = 2$ we have

$$\begin{aligned} u((s_{i_1} s_{i_2})u) &= u(s_{i_2}(s_{i_1}u)) && \text{(c1)} \\ &= u(s_{i_2}(us_{i_1}^{-1})) && \text{induction} \\ &= ((us_{i_2})u)s_{i_1}^{-1} && \text{(m2)} \\ &= s_{i_2}^{-1}s_{i_1}^{-1} && \text{induction} \end{aligned}$$

Now suppose $k \geq 3$ and let $w = s_{i_2} \cdots s_{i_{k-1}}$. Then the following equalities are equivalent:

$$\begin{aligned} u(s_1 w s_k)u &= s_k^{-1} w^{-1} s_1^{-1} \\ u(s_1[u(s_k^{-1} w^{-1})u])u &= s_k^{-1}(u(s_1 w)u) && \text{induction} \\ (us_1)[(u(s_k^{-1} w^{-1})u)u] &= s_k^{-1}[(us_1)(wu)] && \text{(m3)} \\ s_k((us_1)[u(s_k^{-1} w^{-1})]) &= (us_1)(wu) && \text{mult. by } s_k \\ s_k((us_1)[s_k(uw^{-1})]) &= (us_1)(wu) && \text{Lemma 2.4} \\ ([s_k(us_1)]s_k)(uw^{-1}) &= (us_1)(wu) && \text{(m2)} \\ (us_1)(uw^{-1}) &= (us_1)(wu) && \text{Lemma 2.3 (e)} \\ uw^{-1} &= wu && \text{cancel } us_1 \\ wu &= wu && \text{induction} \end{aligned}$$

□

Lemma 2.6. *For all $w_1, w_2 \in T$ we have (c3): $(w_1 u)(w_2 u) = w_2^{-1} w_1$.*

Proof We have

$$\begin{aligned} (w_1 u)(w_2 u) &= (uw_1^{-1})(w_2 u) && \text{by Lemma 2.5} \\ &= u(w_1^{-1} w_2)u && \text{(m3)} \\ &= w_2^{-1} w_1 && \text{by Lemma 2.5.} \end{aligned}$$

□

Lemma 2.7. *For any $w_1, w_2 \in T$ we have (c1) if and only if (c2).*

Proof The following are equivalent:

$$\begin{aligned} w_1(w_2 u) &= (w_2 w_1)u && \text{(c1)} \\ w_1(uw_2^{-1}) &= u(w_1^{-1} w_2^{-1}) && \text{Lemma 2.5} \\ (w_2 u)w_1^{-1} &= (w_2 w_1)u && \text{taking inverses} \end{aligned}$$

The latter equality is (c2). □

Lemma 2.8. *For all $w_1, w_2 \in T$ we have (c2); $(w_1 u)w_2 = (w_1 w_2^{-1})u$.*

Proof The following are equivalent:

$$\begin{aligned}
 (w_1 u)w_2 &= (w_1 w_2^{-1})u && \text{(c2)} \\
 ((w_1 u)w_2)u &= (w_1 w_2^{-1}) && \text{right mult. by } u \\
 w_1(u(w_2 u)) &= (w_1 w_2^{-1}) && \text{(m2)} \\
 w_1 w_2^{-1} &= (w_1 w_2^{-1}) && \text{by Lemma 2.5.}
 \end{aligned}$$

□

Proof (of Theorem 2.1) By Lemmas 2.6, 2.7, and 2.8, L satisfies the Chein relations with respect to u for all elements $g_1, g_2 \in T$. □

Corollary 2.9. *Let T be a group generated by a set $S = \{s_i\}_{i \in J}$ of elements of order two. Here J can be any set. Then, $M(T, u)$ is the Moufang loop generated by $\{s_i\}_{i \in J} \cup \{u\}$ whose defining relations are those of T , together with the Chein relations (c1), (c2), and (c3) where $g_1, g_2 \in \{s_i\}_{i \in J} \cup \{e\}$.*

Corollary 2.10. *Let $(W, \{s_i\}_{i \in I})$ be a Coxeter system with diagram M over some set I . Then, $M(W, u)$ is the Moufang loop generated by $\{s_i\}_{i \in I} \cup \{u\}$ whose defining relations are the Coxeter relations together with the Chein relations (c1), (c2), and (c3) in which $g_1, g_2 \in \{s_i\}_{i \in I} \cup \{e\}$.*

3. THE AUTOMORPHISM GROUP OF $M(G, 2)$

In this section we get a better understanding of the automorphism groups of all the possible Chein loops $M(G, 2)$.

Lemma 3.1. *If $L = M(G, 2)$ for some group G then exactly one of the following holds:*

- (a) L is an elementary abelian 2-group.
- (b) $G \not\cong M(H, 2)$ for any group H .
- (c) $G \cong M(H, 2)$ for an abelian group H where $H \not\cong M(T, 2)$ for any group T .

Proof. If $G \cong M(H, 2)$ for some group H then, in order for G to be a group, H must be abelian. If $H \cong M(T, 2) = \langle T, x \rangle$ for some group T then, since H is abelian, x commutes with all of the elements in T . Thus T is of exponent two. Hence, G and L are elementary abelian 2-groups. □

Lemma 3.2. *If L is a finite elementary abelian 2-group, then $\text{Aut}(L) \cong GL_n(2)$ for some n .*

Lemma 3.3. *Suppose $L = \langle G, u \rangle \cong M(G, 2)$ and $G \not\cong M(H, 2)$ for any group H . Then, any $f \in \text{Aut}(L)$ preserves both G and $L \setminus G$.*

Proof Suppose $G' \neq G$ satisfies $L = G' \uplus G'u' \cong M(G', 2)$ for some $u' \in L$. Then since all elements of $G'u'$ satisfy the Chein relations with respect to the elements in G' we may assume that $u' \in G$. Let $H = G \cap G'$. Then, $G \cap G'u' = Gu' \cap G'u' = Hu'$, so $G = H \uplus Hu'$. It follows that $G \cong M(H, 2)$, a contradiction. \square

Theorem 3.4. *If $L = \langle G, u \rangle \cong M(G, 2)$ and $G \not\cong M(H, 2)$ for any group H then $\text{Aut}(L) \cong G \rtimes \text{Aut}(G)$.*

Proof. We first define the isomorphism

$$(1) \quad \begin{aligned} \Phi: G \rtimes \text{Aut}(G) &\longrightarrow \text{Aut}(L) \\ g\psi &\longmapsto \varphi_g \circ \varphi_\psi. \end{aligned}$$

Here, for $g \in G$ define $\varphi_g: L \longrightarrow L$ by

$$(2) \quad \begin{aligned} \varphi_g(g_1) &= g_1 \\ \text{and } \varphi_g(g_1u) &= (gg_1)u \end{aligned}$$

for any $g_1 \in G$. Moreover, for $\psi \in \text{Aut}(G)$, we define its extension φ_ψ to L by setting

$$(3) \quad \begin{aligned} \varphi_\psi(g) &= \psi(g) \\ \text{and } \varphi_\psi(gu) &= \psi(g)u \end{aligned}$$

for any $g \in G$. It is evident that we then have

$$(4) \quad \varphi_\psi \circ \varphi_g \circ \varphi_\psi^{-1} = \varphi_{\psi(g)}$$

In the sequel we shall simply write ψ for φ_ψ .

We now prove Φ is a well defined isomorphism. Clearly any $f \in \text{Aut}(L)$ is uniquely determined by $f|_G$ and $f(u)$. Since by Lemma 3.3 G is characteristic in L , f preserves G and Gu . Thus we have a natural homomorphism $\pi: \text{Aut}(L) \rightarrow \text{Aut}(G)$ given by $f \mapsto f|_G$. We claim that $\Phi|_{\text{Aut}(G)}$ is well defined and that $\pi \circ \Phi|_{\text{Aut}(G)} = \text{id}$.

Let $\psi \in \text{Aut}(G)$ and denote $\Phi(\psi)$ also by ψ . Since

$$\begin{aligned} \psi((g_1u)(g_2u)) &= \psi(g_2^{-1}g_1) \\ &= \psi(g_2)^{-1}\psi(g_1) \\ &= (\psi(g_1)u)(\psi(g_2)u) \\ &= \psi(g_1u)\psi(g_2u), \end{aligned}$$

$$\begin{aligned}
\psi(g_1(g_2u)) &= \psi((g_2g_1)u) \\
&= \psi(g_2g_1)u \\
&= (\psi(g_2)\psi(g_1))u \\
&= \psi(g_1)(\psi(g_2)u) \\
&= \psi(g_1)\psi(g_2u),
\end{aligned}$$

$$\begin{aligned}
\text{and } \psi((g_1u)g_2) &= \psi((g_1g_2^{-1})u) \\
&= \psi(g_1g_2^{-1})u \\
&= (\psi(g_1)\psi(g_2)^{-1})u \\
&= (\psi(g_1)u)\psi(g_2) \\
&= \psi(g_1u)\psi(g_2)
\end{aligned}$$

for all $g_1, g_2 \in G$, and $\psi^{-1}(gu) = \psi^{-1}(g)u$ for any $g \in G$, $\psi \in \text{Aut}(L)$. This proves the claim on $\Phi|_{\text{Aut}(G)}$. It follows that

$$K = \Phi(\text{Aut}(G)) \cong \text{Aut}(G)$$

is a complement to $N = \ker \pi$, so that $\text{Aut}(L) \cong N \rtimes K$.

We now examine N . We define a map $\chi: N \rightarrow G$ by $f(u) = \chi(f)u$. This is a homomorphism because

$$(5) \quad (f \circ g)(u) = f(\chi(g)u) = \chi(g)f(u) = \chi(g)(\chi(f)u) = (\chi(f)\chi(g))u.$$

Clearly any $f \in \ker(\chi) \leq N$ fixes u as well as all elements in G , so χ is injective. We claim that $\Phi|_G$ is well-defined and $\chi \circ \Phi|_G = \text{id}$. Let $g \in G$ and write $\Phi(g) = \varphi_g$. Since

$$\begin{aligned}
\varphi_g((g_1u)(g_2u)) &= \varphi_g(g_2^{-1}g_1) \\
&= g_2^{-1}g^{-1}gg_1 \\
&= ((gg_1)u)((gg_2)u) \\
&= \varphi_g(g_1u)\varphi_g(g_2u),
\end{aligned}$$

$$\begin{aligned}
\varphi_g(g_1(g_2u)) &= \varphi_g((g_2g_1)u) \\
&= (gg_2g_1)u \\
&= g_1((gg_2)u) \\
&= \varphi_g(g_1)\varphi_g(g_2u),
\end{aligned}$$

$$\begin{aligned}
\text{and } \varphi_g((g_1u)g_2) &= \varphi_g((g_1g_2^{-1})u) \\
&= (gg_1g_2^{-1})u \\
&= ((gg_1)u)g_2 \\
&= \varphi_g(g_1u)\varphi_g(g_2)
\end{aligned}$$

for all $g_1, g_2 \in G$, and $\varphi_g^{-1} = \varphi_{g^{-1}}$, $\varphi_g \in \text{Aut}(L)$. Clearly $\chi \circ \Phi(g) = g$. This proves the claim on $\Phi|_G$. Thus

$$N = \Phi(G) = \{\varphi_g \mid g \in G\} \cong G.$$

We conclude that $\text{Aut}(L) \cong N \rtimes K \cong G \rtimes \text{Aut}(G)$. \square

Theorem 3.5. *If $L = \langle G, u_2 \rangle \cong M(G, 2)$ for a nonabelian group $G = \langle H, u_1 \rangle \cong M(H, 2)$ then $\text{Aut}(L) \cong (H \times H) \rtimes (S_3 \times \text{Aut}(H))$.*

Proof. First notice that $L = H \uplus Hu_1 \uplus Gu_2 = H \uplus Hu_1 \uplus Hu_2 \uplus Hu_3$, where $\{1, u_1, u_2, u_3\}$ forms a Klein four group. Clearly any $f \in \text{Aut}(L)$ is uniquely determined by its action on H , u_1 and u_2 . Since H is abelian, but $H \not\cong M(T, 2)$ for any group T , L contains elements of order ≥ 2 and they are all in H . Let h be any such element, then $H = C_L(h)$, so H is characteristic in L and $f|_H \in \text{Aut}(H)$. Thus f fixes H , with restriction in $\text{Aut}(H)$, while permuting the remaining three cosets of H . Thus we have a homomorphism $\pi: \text{Aut}(L) \rightarrow S_3 \times \text{Aut}(H)$. We now show that π is surjective and that $N = \ker \pi$ has a complement $K \cong S_3 \times \text{Aut}(H)$ so that $\text{Aut}(L) \cong N \rtimes K$.

For $i = 1, 2$, let $\Psi_i: H \rtimes \text{Aut}(H) \rightarrow \text{Aut}(G_i)$ be the injective homomorphism defined in (1), viewing $G_i \cong M(H, 2)$. Likewise, for $i = 1, 2$, let $G_i = \langle H, u_i \rangle \cong M(H, 2) \cong G$ and let $\Phi_i: G_i \rtimes \text{Aut}(G_i) \rightarrow \text{Aut}(L)$ be as in (1), viewing $L = M(G_i, 2) = \langle G_i, u_{3-i} \rangle$.

Let

$$A = (\Phi_1 \circ \Psi_1)(\text{Aut}(H)).$$

To see what this group is, let $\psi \in \text{Aut}(H)$ and denote $\psi = \Phi_1 \circ \Psi_1(\psi)$. Then, $\psi(hu_i) = \psi(h)u_i$, for $i = 1, 2, 3$ and all $h \in H$.

For $i = 1, 2$, set $\sigma_i = \Phi_i(u_i) \in \text{Aut}(L)$; recall $u_i \in G_i$. Then, σ_i fixes $G_i = H \uplus Hu_i$ elementwise, while, for any $h \in H$ we have

$$\begin{aligned}
hu_{3-i} &\mapsto (u_i h)u_{3-i} = h(u_i u_{3-i}) = hu_3 \\
hu_3 &\mapsto (u_i h)(u_i u_{3-i}) = (u_i^2 h)u_{3-i} = hu_{3-i}.
\end{aligned}$$

Thus $S = \langle \sigma_1, \sigma_2 \rangle \cong S_3$ is a subgroup of $\text{Aut}(L)$ mapping onto $S_3 \times \{1\}$ under π . Clearly any $\psi \in A$ commutes with any element of S . Thus

$$K = \langle S, A \rangle \cong S_3 \times \text{Aut}(H)$$

is a complement to $N = \ker \pi$.

We now show that $N \cong H \times H$. Define a map $\chi: N \rightarrow H \times H$ by sending $f \mapsto (\chi_1(f), \chi_2(f))$, where $f(u_i) = \chi_i(f)u_i$ for $i = 1, 2$. As in (5), this is a homomorphism. Clearly χ is injective because $f \in \ker \chi \leq N$ fixes each element of H , as well as each u_i for $i = 1, 2, 3$. Finally we notice that χ is onto.

Fix $i \in \{1, 2\}$ and $h \in H$. Then, for $h' \in H$ we have $\Psi_i(h)(h') = h'$ and $\Psi_i(h)(h'u_i) = (hh')u_i$. Moreover, $(\Phi_i \circ \Psi_i)(h)$, extends $\Psi_i(h) \in \text{Aut}(G_i)$ to L as $gu_{3-i} \mapsto (\Psi_i(h)(g))u_{3-i}$, for all $g \in G_i$. Summarizing, $\Psi_i(h)$ is given by

$$(6) \quad \begin{aligned} h' &\mapsto h' & (h'u_i) &\mapsto (hh')u_i & h'u_{3-i} &\mapsto h'u_{3-i} & h'u_3 &\mapsto (h'h^{-1})u_3 \end{aligned}$$

for any $h' \in H$. Hence, if, for $h_1, h_2 \in H$ we set $f = ((\Phi_1 \circ \Psi_1)(h_1)) \circ ((\Phi_2 \circ \Psi_2)(h_2)) \in \text{Aut}(L)$, then, $f(u_i) = h_i u_i$, so $(\chi_1(f), \chi_2(f)) = (h_1, h_2)$. We conclude that $N \cong H \times H$ and $\text{Aut}(L) \cong (H \times H) \rtimes (S \times \text{Aut}(H))$. \square

4. AMALGAMS

Let $(W, \{s_i\}_{i \in I})$ be a Coxeter system with diagram \mathbf{M} over I and let $L = M(W, 2)$ be the associated Coxeter-Chein loop. Let $\mathbf{I} = I \cup \{\infty\}$ and set $s_\infty = u$.

The Coxeter-Chein presentation can be interpreted as a simplicial amalgam in the sense of [1]. We first construct a simplicial complex from the graph underlying \mathbf{M} .

Definition 4.1. Let \mathbf{M} be a Coxeter diagram over I associated with Coxeter matrix $(m_{ij})_{i,j \in I}$. We call $\Delta(\mathbf{M}) = (I, E)$ the *graph underlying* \mathbf{M} , where $E = \{\{i, j\} \in I \times I \mid m_{ij} \geq 3\}$.

From now on let $\Delta = (I, E)$ be the graph underlying \mathbf{M} .

Definition 4.2. A *simplicial complex* is a pair $\mathbf{E} = (E, F)$, where E is a set and F is a collection of subsets of E , called *simplices*, such that $\{e\} \in F$ for each $e \in E$, and if $\tau \in F$ and ρ is a subset of τ , then $\rho \in F$; in this case we say that ρ is a *face* of τ and we write $\rho \leq \tau$. A simplex σ with $|\sigma| = r + 1$ is called an *r-simplex*.

Definition 4.3. Given a graph $\Delta = (I, E)$ we define its associated simplicial complex as $\mathbf{E}(\Delta) = (E, F)$, where F is the collection of subsets of cardinality at most 3 of E .

We now let $\mathbf{E} = (E, F)$ be the simplicial complex associated to Δ .

Definition 4.4. A *simplicial amalgam of Moufang loops* over a simplicial complex $\mathbf{E} = (E, F)$ is a collection of Moufang loops with connecting homomorphisms,

$$\mathcal{G} = \{G_\sigma, \psi_\tau^\rho \mid \rho, \sigma, \tau \in F, \rho \leq \tau\},$$

where, for all $\rho, \tau \in F$ with $\rho \leq \tau$, we have an injective homomorphism

$$\psi_\tau^\rho: G_\tau \hookrightarrow G_\rho.$$

We require that, whenever $\rho \leq \sigma \leq \tau$, then $\psi_\tau^\rho = \psi_\sigma^\rho \circ \psi_\tau^\sigma$.

A *completion* of \mathcal{G} is a Moufang loop G together with a collection $\phi = \{\phi_\sigma \mid \sigma \in F\}$ of homomorphisms $\phi_\sigma: G_\sigma \rightarrow G$, such that whenever $\sigma \leq \tau$, we have $\phi_\sigma \circ \psi_\tau^\sigma = \phi_\tau$. The amalgam \mathcal{G} is *non-collapsing* if it has a non-trivial completion. A completion $(\hat{G}, \hat{\phi})$ is called *universal* if for any completion (G, ϕ) there is a (necessarily unique) surjective loop homomorphism $\pi: \hat{G} \rightarrow G$ such that $\phi = \pi \circ \hat{\phi}$.

Remark 4.5. Definition 4.4 is a specialization of the simplicial amalgams in concrete categories defined in [1]

We shall now define a simplicial amalgam \mathcal{G} of Moufang loops as in Definition 4.4 over the simplicial complex $\mathbf{E} = (E, F)$. For any subset $J \subseteq I$, we let

$$L_J = \langle s_j, s_\infty \mid j \in J \rangle_L \cong M(W_J, 2).$$

Then, for $\sigma \in F$ we let G_σ be a copy of L_J , where $J = \bigcap_{e \in \sigma} e$. Moreover, for all $\rho, \tau \in F$ with $\rho \leq \tau$, $\psi_\tau^\rho: G_\tau \hookrightarrow G_\rho$ is given by natural inclusion of subloops in L .

Lemma 4.6. L is a completion of \mathcal{G} .

Proof In Definition 4.4 take the collection $\phi = \{\phi_\sigma = \text{id} \mid \sigma \in F\}$. \square

Definition 4.7. An amalgam of type \mathcal{G} is an amalgam $\mathcal{G}' = \{G_\sigma, \varphi_\tau^\rho \mid \rho, \sigma, \tau \in F, \rho \leq \tau\}$, such that, for each $\sigma \in F$, the loop G_σ is that of \mathcal{G} , and, for each $\rho, \tau \in F$ with $\rho \leq \tau$, the images of the connecting maps φ_τ^ρ and ψ_τ^ρ coincide.

Our aim is now to classify all amalgams of Moufang loops of type \mathcal{G} using 1-cohomology on \mathbf{E} as described in [1].

Definition 4.8. Let $\mathcal{G} = (G_\bullet, \psi_\bullet)$ be an amalgam over the simplicial complex $\mathbf{E} = (E, F)$. The *coefficient system associated to \mathcal{G}* is the collection

$$\mathcal{A} = \{A_\sigma, \alpha_\rho^\sigma \mid \sigma < \rho \text{ with } \sigma, \rho \in F\},$$

where, for each $\sigma \in F$, we let

$$A_\sigma = \bigcap_{\rho > \sigma} \text{Stab}_{\text{Aut}(G_\sigma)}(G_\rho),$$

and, for each $\sigma, \rho \in F$ with $\sigma < \rho$, we have a homomorphism

$$\begin{aligned} \alpha_\rho^\sigma: A_\sigma &\rightarrow A_\rho \\ f &\mapsto f|_{G_\rho} \end{aligned}$$

If $\sigma = \{e_1, \dots, e_l\}$ we shall sometimes write A_{e_1, \dots, e_l} for A_σ . As for amalgams, we shall use the shorthand notation $\mathcal{A} = \{A_\bullet, \alpha_\bullet\}$.

Remark 4.9. Strictly speaking, one should define $\alpha_\rho^\sigma(f)$ as $(\psi_\rho^\sigma)^{-1} \circ f \circ \psi_\rho^\sigma$, where, $(\psi_\rho^\sigma)^{-1}$ is understood to be defined only on the image of ψ_ρ^σ . However, since the map ψ_ρ^σ is given by the inclusion $G_\rho \subseteq G_\sigma$ of subloops of L , we have chosen to describe them as above for better readability.

From now on the coefficient system associated to \mathcal{G} defined above will be

$$\mathcal{A} = \{A_\sigma, \alpha_\rho^\sigma \mid \sigma < \rho \text{ with } \sigma, \rho \in F\}.$$

We now compute the groups and connecting maps of \mathcal{A} .

Lemma 4.10. *Let $e, f, g \in E$ be distinct and let $\sigma \in F$. Then,*

- (a) *If $\bigcap_{e \in \sigma} e = \emptyset$, then $A_\sigma = \{\text{id}\}$.*
- (b) *If $e \cap f = \{j\}$, then $A_{e,f} = \langle \gamma_j \rangle \cong \mathbb{Z}_2$, where $\gamma_j: s_j \leftrightarrow s_j s_\infty$;*
- (c) *If $e = \{i, j\}$ with $m_{ij} \geq 3$, then $A_e = \langle \gamma_{ij} \rangle \cong \mathbb{Z}_2$, where*

$$\gamma_{ij}: \begin{cases} s_\infty \text{ is fixed} \\ s_i \leftrightarrow s_i s_\infty \\ s_j \leftrightarrow s_j s_\infty \\ s_i s_j \leftrightarrow s_j s_i \end{cases}.$$

Thus, if $f = \{j, k\}$, then $\alpha_{e,f}^e: \gamma_{ij} \mapsto \gamma_j$;

- (d) *If $e \cap f \cap g = \{j\}$, then $A_{e,f,g} = \langle \gamma_j \rangle \cong \mathbb{Z}_2$, where $\gamma_j: s_j \leftrightarrow s_j s_\infty$.*

In this case $\alpha_{e,f,g}^{e,f}: \gamma_j \mapsto \gamma_j$.

Proof (a) This is clear since G_σ has only one non-trivial element.

(b) We have $G_{\{i,j\},\{j,k\}} = L_j \cong 2^2$. Thus $\text{Aut}(L_j) = \text{Sym}(\{s_j, s_\infty, s_j s_\infty\})$. The elements of $A_{\{i,j\},\{j,k\}}$ also preserve $G_{\{i,j\},\{j,k\},\{i,k\}} = L_\emptyset = \langle s_\infty \rangle_L$ and so they must permute $\{s_j, s_j s_\infty\}$. Call the non-trivial automorphism γ_j .

(c) Assume $m_{ij} \geq 3$. Let γ be a non-trivial element in $A_{\{i,j\}}$. Then γ is an automorphism of L_{ij} preserving L_i , L_j and L_\emptyset . Setting $G = W_{\{i,j\}}$ and $H = \langle s_i s_j \rangle$ it follows from Theorem 3.5 that H is characteristic in L . Thus any automorphism of L_{ij} permutes the three copies of G in L_{ij} : $C_i = \langle H, s_i \rangle = \langle H, s_j \rangle$, $C_\infty = \langle H, s_\infty \rangle$, and $C_{i,\infty} = \langle H, s_i s_\infty \rangle =$

$\langle H, s_j s_\infty \rangle$. The automorphism γ therefore simultaneously permutes the intersections: $C_i \cap L_i = \langle s_i \rangle$, $C_\infty \cap L_i = \langle s_\infty \rangle$, and $C_{i,\infty} \cap L_i = \langle s_i s_\infty \rangle$ as well as those with L_j . Thus, since s_∞ is fixed, γ must interchange s_i and $s_i s_\infty$ as well as s_j and $s_j s_\infty$. It then follows that $s_i s_j \mapsto (s_i s_\infty)(s_j s_\infty) = (s_\infty s_i)(s_j s_\infty) = s_\infty(s_i s_j)s_\infty = s_j s_i$, so γ acts by sending every element of H to its inverse. (d) Immediate from (b). \square

Remark 4.11. Lemma 4.10 can be summarized as saying that

$$A_\sigma \cong \begin{cases} \mathbb{F}_2 & \text{if } \bigcap_{e \in \sigma} \sigma \neq \emptyset \\ \{0\} & \text{else} \end{cases}$$

and, for $\sigma < \rho$, α_ρ^σ is an isomorphism if and only if $\bigcap_{e \in \rho} e \neq \emptyset$ or $\bigcap_{e \in \sigma} e = \emptyset$.

It follows from Remark 4.11 that the cochain complex of pointed sets obtained from the coefficient system $\mathcal{A} = \{A_\bullet, \alpha_\bullet\}$ in [1] can be constructed directly from the underlying graph Δ of M .

Definition 4.12. Given a graph $\Delta = (I, E)$ with associated simplicial complex $\mathbf{E} = (E, F)$, we define $F_\bullet^r = \{\sigma \in F^r \mid \bigcap_{e \in \sigma} e \neq \emptyset\}$ and $F_\bullet = \bigcup_{r \in \mathbb{N}} F_\bullet^r$. We then define a cochain complex $\mathcal{C}^\bullet(\Delta)$ of \mathbb{F}_2 -vector spaces:

$$\mathcal{C}^0 \xrightarrow{d^0} \mathcal{C}^1 \xrightarrow{d^1} \mathcal{C}^2$$

where \mathcal{C}^i is an \mathbb{F}_2 -vector space with formal basis $\{a_\sigma \mid \sigma \in F_\bullet^i\}$ and, for $r = 0, 1$, the coboundary map d^r is \mathbb{F}_2 -linear and given by

$$d^r : a_\sigma \mapsto \sum_{\tau \in F_\bullet^{r+1} : \sigma < \tau} a_\tau.$$

Lemma 4.13. *The cochain complex of pointed sets associated to the coefficient system $\mathcal{A} = \{A_\bullet, \alpha_\bullet\}$ as defined in [1] is $\mathcal{C}^\bullet(\Delta)$.*

From now on let $\mathcal{C}^\bullet = \mathcal{C}^\bullet(\Delta)$.

Definition 4.14. Given a graph Δ and $r = 0, 1$, we define

$$\begin{aligned} Z^r(\Delta) &= \ker d^r && r\text{-cocycle group} \\ B^{r+1}(\Delta) &= \text{im } d^r && (r+1)\text{-coboundary group} \\ H^r(\Delta) &= Z^r(\Delta)/B^r(\Delta) && r\text{-cohomology group} \end{aligned}$$

We denote the cohomology class of $z \in Z^r(\Delta)$ as $[z] = z + B^r(\Delta)$. Here d^r is the r -coboundary map of the cochain complex $\mathcal{C}^\bullet(\Delta)$.

The motivation to compute these groups is the following.

Theorem 4.15. (cf. Theorem 2 of [1]) *The isomorphism classes of amalgams of type \mathcal{G} are bijectively parametrized by elements of $H^1(\Delta)$.*

To make this parametrization explicit, we need a concrete basis for $H^1(\Delta)$.

A basis for $H^1(\Delta)$. We consider the cochain complex restricted to the set of edges on a vertex.

Definition 4.16. Let $\Delta = (I, E)$ be a graph and let $i \in I$. Let $\Delta_i = (I_i, E_i)$ be the subgraph on the set of edges incident to i . Write $v(i) = |E_i|$.

For any $i \in I$, let $\mathbf{E}_i = (E_i, F_i)$ denote the simplicial complex $\mathbf{E}(\Delta_i)$. Since $i \in \bigcap_{e \in \sigma} e$ for all $\sigma \in F_i$, Δ_i is the full simplex on E_i .

Definition 4.17. Given a graph $\Delta = (I, E)$ and $i \in I$, define $\mathcal{C}_i^\bullet(\Delta)$ to be the cochain subcomplex of $\mathcal{C}^\bullet(\Delta)$ associated to Δ_i :

$$\mathcal{C}_i^0 \xrightarrow{d_i^0} \mathcal{C}_i^1 \xrightarrow{d_i^1} \mathcal{C}_i^2$$

That is, for each $r = 0, 1, 2$, \mathcal{C}_i^r is the subspace of \mathcal{C}^r spanned by $\{a_\sigma \mid \sigma \in F_i^r\}$ and coboundary maps are given by:

$$\begin{aligned} d_i^0: \mathcal{C}_i^0 &\rightarrow \mathcal{C}_i^1 \\ a_\sigma &\mapsto \sum_{\tau \in F_i^1: \sigma < \tau} a_\tau. \end{aligned} \quad \text{and} \quad \begin{aligned} d_i^1: \mathcal{C}_i^1 &\rightarrow \mathcal{C}_i^2 \\ a_\sigma &\mapsto \sum_{\tau \in F_i^2: \sigma < \tau} a_\tau. \end{aligned}$$

Note that $\mathcal{C}_i^\bullet(\Delta)$ is naturally isomorphic to $\mathcal{C}^\bullet(\Delta_i)$.

Lemma 4.18.

(a)

$$\begin{aligned} \mathcal{C}^1 &= \bigoplus_{i \in I} \mathcal{C}_i^1 = \bigoplus_{i \in I: v(i) \geq 2} \mathcal{C}_i^1, \\ \mathcal{C}^2 &= \bigoplus_{i \in I} \mathcal{C}_i^2 = \bigoplus_{i \in I: v(i) \geq 3} \mathcal{C}_i^2. \end{aligned}$$

(b) For $r = 0, 1$ and $i \in I$, we have

$$\begin{aligned} d_i^r: \mathcal{C}_i^r &= \{0\} \rightarrow \mathcal{C}_i^{r+1} = \{0\} & \text{if } v(i) \leq r \\ d_i^r: \mathcal{C}_i^r &= \langle a_{E_i} \rangle \rightarrow \mathcal{C}_i^{r+1} = \{0\} & \text{if } v(i) = r + 1 \end{aligned}$$

(c) $d^1 = \sum_{i \in I: v(i) \geq 2} d_i^1$.

(d) $Z^1(\Delta) = \bigoplus_{i \in I: v(i) = 2} \mathcal{C}_i^1 \oplus \bigoplus_{i \in I: v(i) \geq 3} Z^1(\Delta_i)$.

Proof Immediate from Definitions 4.12 and 4.17. \square

Lemma 4.19. Suppose $\Delta = (I, E)$ is a star with $E = \{e_0, \dots, e_{v-1}\}$.

((a)) $Z = \{d^0(a_{e_i}) \mid 0 < i \leq v-1\}$ is a basis for $B^1(\Delta) = Z^1(\Delta)$.

((b)) $B = \{d^1(a_{e_i, e_j}) \mid 0 < i < j \leq v-1\}$ is a basis for $B^2(\Delta)$.

Proof (a) That Z is linearly independent follows from the fact that $d^0(a_{e_i})$ is the only element in Z with non-zero coefficient for a_{e_i, e_0} . Now note that

$$\begin{aligned} \sum_{i=0}^{v-1} d^0(a_{e_i}) &= \sum_{i=0}^{v-1} \sum_{j \neq i} a_{e_i, e_j} \\ &= \sum_{\{i, j\} \subseteq \{0, 1, \dots, v-1\}} 2a_{e_i, e_j} = 0. \end{aligned}$$

Hence, $d^0(a_{\{0\}}) \in \langle Z \rangle$. That Z is also a basis for $\ker d^1$ will be proved below. (b) We compute $\dim \ker d^1 = \dim \mathcal{C}^1 - \dim \operatorname{im} d^1$. Clearly $\dim \mathcal{C}^1 = \binom{v}{2}$. If $v = 1$, then $\dim \mathcal{C}^1 = 0 = \dim \ker d^1 = v - 1$. If $v = 2$, then clearly $\dim \operatorname{im} d^1 = \dim \mathcal{C}^2 = 0$ and so $\dim \ker d^1 = 1 - 0 = v - 1$. If $v = 3$, then $\dim \operatorname{im} d^1 = \dim \mathcal{C}^2 = 1$ and so $\dim \ker d^1 = 3 - 1 = v - 1$. Suppose now that $v \geq 4$. Then, the set $B = \{d^1(a_{e_i, e_j}) \mid 1 \leq i < j \leq v - 1\}$ is linearly independent because $d^1(a_{e_i, e_j})$ is the unique vector in B with non-zero coefficient for a_{e_0, e_i, e_j} . So $\dim \operatorname{im} d^1 \geq \binom{v-1}{2}$. Note however that for any $i \in \{1, \dots, v - 1\}$,

$$\begin{aligned} \sum_{j \neq i} d^1(a_{e_i, e_j}) &= \sum_{j \neq i} \sum_{k \neq i, j} a_{e_i, e_j, e_k} \\ &= \sum_{\{j, k\} \subseteq \{0, \dots, \hat{i}, \dots, v-1\}} 2a_{e_i, e_j, e_k} = 0. \end{aligned}$$

Hence $d^1(a_{e_i, e_0}) = \sum_{j \neq i, 0} d^1(a_{e_i, e_j}) \in \langle Z \rangle$. Thus, $\dim \operatorname{im} d^1 \leq \binom{v-1}{2}$. It follows that

$$\dim \ker d^1 = \binom{v}{2} - \binom{v-1}{2} = v - 1.$$

To see that Z is a basis for $\ker d^1$ note that $d^1 \circ d^0 = 0$ since for each $i \in V$ we have $d^1 \circ d^0(a_{e_i}) = \sum_{j \neq i} d^1(a_{e_i, e_j})$ and this was shown to equal 0. \square

Lemma 4.20. ((a)) *We have $\dim_{\mathbb{F}_2}(Z^1(\Delta)) = \sum_{i \in I} (v(i) - 1) = 2|E| - |I|$. More precisely, for each $i \in I$, pick any $f_i \in E_i$. Then $Z^1(\Delta)$ has basis*

$$Z_\Delta = \{d_i^0(a_e) \mid i \in I \text{ with } v(i) \geq 2 \text{ and } e \in E_i - \{f_i\}\}.$$

((b)) *We have $\dim_{\mathbb{F}_2} B^1(\Delta) = |E| - 1$. More precisely, given any $e_0 \in E$, it has \mathbb{F}_2 -basis*

$$B_\Delta = \{d^0(a_e) \mid e \in E - \{e_0\}\}.$$

Proof (a) This follows by combining Lemma 4.18 part (d) and Lemma 4.19.

(b) Suppose that we have

$$(7) \quad \sum_{e \in E} \lambda_e d^0(a_e) = 0 \text{ with } \lambda_{e_0} = 0.$$

From Lemma 4.19 we derive the following observation: For each $i \in I$ and $f_i \in E_i$,

(8) the set $\{d_i^0(a_e) \mid e \in E_i - \{f_i\}\}$ is linearly independent in $Z^1(\Delta)$,

$$(9) \quad d_i^0(a_{f_i}) = \sum_{e \in E_i - \{f_i\}} d_i^0(a_e).$$

Now it follows from (7), 8 and 9, that if for some $i \in I$ and $f_i \in E_i$ we have $\lambda_{f_i} = 0$, then also $\lambda_e = 0$ for all $e \in E_i - \{f_i\}$. Since Δ is connected and we have $\lambda_{e_0} = 0$, it follows that $\lambda_e = 0$ for all $e \in E$. Also note that

$$\sum_{e \in E} d^0(a_e) = \sum_{i \in I: v(i) \geq 2} \sum_{e \in E_i} d_i^0(a_e) = 0,$$

so $d^0(a_{e_0}) \in \langle d^0(a_e) \mid e \in E - \{e_0\} \rangle$. Thus, B_Δ is linearly independent and spans $B^1(\Delta)$. \square

Let T be a spanning tree for Δ with edge set $E(\mathsf{T}) = E - \{e_1, \dots, e_n\}$ and, for $j = 1, \dots, n$, let $e_j = \{o_j, t_j\}$.

Proposition 4.21. *We have*

$$\dim_{\mathbb{F}_2} H^1(\Delta) = |E| - |I| + 1 = n.$$

More precisely, $H^1(\Delta)$ has a basis

$$H_\Delta = \{[d_{o_j}^0(e_j)] \mid j = 1, 2, \dots, n\}.$$

Proof From Lemma 4.20, we find

$$\begin{aligned} \dim_{\mathbb{F}_2} H^1(\Delta) &= \dim_{\mathbb{F}_2} Z^1(\Delta) - \dim_{\mathbb{F}_2} B^1(\Delta) \\ &= 2|E| - |I| - (|E| - 1) \\ &= |E| - |I| + 1. \end{aligned}$$

Recalling that T is a tree on the vertex set I , we find $|E| - |I| + 1 = (|E(\mathsf{T})| - |I| + 1) + n = 0 + n$. To describe a basis for $H^1(\Delta)$, define a tree

$$\overline{\mathsf{T}} = (I \cup \{\overline{t_j}, \overline{o_j}\}, E(\mathsf{T}) \cup \{\overline{e_j}, \overline{e_j'}\}_{j=1}^n),$$

where $\overline{e_j} = \{o_j, \overline{t_j}\}$ and $\overline{e_j'} = \{\overline{o_j}, t_j\}$ for each $j = 1, 2, \dots, n$ (we double each edge e_j and attach one copy to each of its original endpoints). The map $\kappa: \overline{\mathsf{T}} \rightarrow \Delta$ given by

$$\kappa(i) = \begin{cases} i & \text{if } i \in I \\ t_j & \text{if } i = \overline{t_j} \\ o_j & \text{if } i = \overline{o_j} \end{cases}$$

induces an isomorphism $\kappa: Z^1(\overline{\mathsf{T}}) \rightarrow Z^1(\Delta)$ since valency of the vertices in I is preserved and the valency of $\overline{t_j}$ and $\overline{o_j}$ is 1. On the other

hand $\kappa: B^1(\overline{\mathbb{T}}) \rightarrow B^1(\Delta)$ sends both $d^0(\overline{e_j}) = d_{o_j}^0(\overline{e_j})$ and $d^0(\overline{e_j}') = d_{t_j}^0(\overline{e_j}')$ to $d^0(e_j) = d_{o_j}^0(e_j) + d_{t_j}^0(e_j)$. So, H_Δ is a basis for $H^1(\Delta) = Z^1(\Delta)/B^1(\Delta)$. \square

Description of the amalgams.

Theorem 4.22. (Classification of Loops of type \mathcal{G}) *Let $\mathbb{T} = (I, E(\mathbb{T}))$ be a spanning tree for Δ and let $E - E(\mathbb{T}) = \{e_1, \dots, e_n\}$. For each $j \in \{1, \dots, n\}$ pick a vertex $o_j \in e_j$. There is a bijection between amalgams*

$$\mathcal{L}^\delta = \{L_i, L_e, \varphi_i^e \mid i \in I, i \in e \in E, \varphi_i^e: L_i \hookrightarrow L_e\}$$

of type \mathcal{G} and subsets $\delta \subseteq \{1, 2, \dots, n\}$ given as follows: For every vertex $i \in I$ and edge e with $i \in e$, we have

$$(10) \quad \varphi_i^e = \begin{cases} s_i \mapsto s_i s_\infty & \text{if } i = o_j \text{ and } e = e_j \text{ for some } j \in \delta \\ s_\infty \mapsto s_\infty & \\ \psi_i^e = \text{id} & \text{else} \end{cases}$$

In particular, for $\delta = \emptyset$, we recover \mathcal{G} .

Proof We shall use the parametrization alluded to in Theorem 4.15. Using Proposition 4.21 we see that the subsets $\delta \subseteq \{1, 2, \dots, n\}$ bijectively parametrize the elements in the \mathbb{F}_2 -vector space $H^1(\Delta)$ as follows:

$$\delta \mapsto \left[z = \sum_{j \in \delta} d_{o_j}^0(e_j) \right].$$

Following [1, §4], we shall use the cocycle $[z]$ to construct the corresponding normalized amalgam

$$\mathcal{G}^{[z]} = \{G_\sigma, \varphi_\tau^\rho \mid \rho, \sigma, \tau \in F, \rho \leq \tau\}.$$

Choose a total order \prec on E such that $e_j \prec f$ for all $f \in E_{o_j} \cap E(\mathbb{T})$ (we can even achieve $e_j \prec f$ for all $f \in E(\mathbb{T})$). For $\rho, \tau \in F$ with $\rho \leq \tau$, we define φ_τ^ρ as follows (here $e = \max_\prec \rho$ and $f = \max_\prec \tau$ so $e \preceq f$):

$$(11) \quad \varphi_\tau^\rho = \begin{cases} \psi_\tau^\rho & \text{if } e = f \\ (\psi_\rho^{\{e\}})^{-1} \circ \psi_{\{e,f\}}^{\{e\}} z_{\{e,f\}}^{-1} \circ \psi_\tau^{\{e,f\}} & \text{if } e \prec f, \end{cases}$$

where $z = \sum_{\{e,f\} \in F_\bullet^1} z_{\{e,f\}}$ and $z_{\{e,f\}} = \zeta_{e,f} a_{\{e,f\}}$, for some $\zeta_{e,f} \in \mathbb{F}_2$. In particular, the amalgam is uniquely determined by the pairs $\rho \in F^0$, $\tau \in F_\bullet^1$ with $\rho \leq \tau$. To compute $\zeta_{e,f}$, recall that for $j = 1, 2, \dots, n$, $d_{o_j}^0(e_j) = \sum_{f \in E_{o_j} - \{e_j\}} a_{\{e_j, f\}}$ and so

$$(12) \quad \zeta_{e,f} = \begin{cases} 1 & \text{if } j \in \delta, o_j \in e \cap f \text{ and } |\{e, f\} \cap \{e_1, \dots, e_n\}| = 1 \\ 0 & \text{else.} \end{cases}$$

To view the resulting amalgam in the form presented in the theorem, first note that, for any j , $a_{\{e_j, f\}}$ acts as $\gamma_{o_j} : s_{o_j} \leftrightarrow s_{o_j} s_\infty$. Now let $i \in I$ and $i \in e, f \in E$. If $i \in I - \{o_1, \dots, o_n\}$, then $\varphi_{e, f}^e = \psi_i^e$ is natural inclusion. Moreover if $f, g \in E(\mathbb{T}) \cap E_{o_j}$, then $e_j \prec f$ so that $\varphi_{e_j, f}^{e_j} = \gamma_{o_j}$ and $\varphi_{e_j, f}^f = \psi_{o_j}^f$, and $\varphi_{f, g}^f = \psi_{o_j}^f$ are inclusion. However, if $o_j = o_k$, then $\varphi_{e_j, e_k}^{e_j}$ and $\varphi_{e_j, e_k}^{e_k}$ are *both* the identity, whereas (10) insists both are twisted by γ_{o_j} . Replacing L_{e_j, e_k} by $L_{o_j}^{\gamma_{o_j}}$ induces an isomorphism between the resulting amalgams. \square

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